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Анализ свободных колебаний стержневых систем с помощью метода конечных элементов в усилиях с расширенными функциями формы

Хыу Хиеу Нго[1]✉, Владимир Владимирович Лалин[1], Ты Куанг Чунг Ле[2]

[1]Санкт-Петербургский политехнический университет Петра Великого, Санкт-Петербург, Российская Федерация
[2]Вьетнамский институт атомной энергии, Ханой, Вьетнам
✉ hieupolytech1993@gmail.com

Аннотация. Для анализа свободных колебаний стержневых систем предложена формулировка метода конечных элементов в усилиях с расширенными функциями формы, полученными с использованием форм колебаний неподвижно закрепленного стержня. Приведены примеры, иллюстрирующие точность численного решения для трех типов конечных элементов. Использование предложенного подхода для пространственной рамы дало различия между численными решениями для восьми конечных элементов и точными результатами менее 5% для первых пяти собственных частот.
Ключевые слова: метод конечных элементов в усилиях, расширенные функции формы, метод сил, матрица податливости, обратная матрица масс
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CONSTRUCTION MECHANICS

Original article

Free vibration analysis of rod systems using the finite element force method with extended shape functions

Hieu H. Ngo[1]✉, Vladimir V. Lalin[1], Trung T.Q. Le[2]

[1]Peter the Great St. Petersburg Polytechnic University, Saint Petersburg, Russian Federation
[2]Vietnam Atomic Energy Institute, Hanoi, Vietnam
✉ hieupolytech1993@gmail.com

Abstract. For free vibration analysis of rod systems formulation of the finite element force method with extended shape functions obtained by using the mode shapes of a fixed-fixed rod is proposed. Examples illustrating the accuracy of numerical solutions for three types of finite elements are given. The use of proposed approach for the space frame gave differences between numerical solutions dealt with 8 finite elements and exact results less than 5% for the first five natural frequencies.
***Keywords*:** finite element force method, extended shape functions, force method, flexibility matrix, inverse mass matrix
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Introduction

In architecture and construction, rod systems play a significant role and are widely used for various structures, including airport roofs, bridges, electric poles, and many others [1,2]. For structural analysis and design, the displacement method has been traditionally more commonly used. However, in recent decades, there has been significant development and research on the force method. The process of developing the force method can be divided into two main stages:

Stage 1: Performing structural analysis calculations manually by hand. From the beginning of the 19th century, J.A. Eytelwein[1], C.L.M. Navier[2], B.P.E. Clapeyron[3], J.A.C. Bress[4] developed the system of force equilibrium equations for a statically determinate beam system (or a statically inde-terminate beam system using Maxwell's theorem). The main unknowns in this analysis are the forces. This approach is known by different names, for example, the method of consistent deformation, unit load method, flexibility method, and the superposition equations method, but most often it is known as the force method [3].

Stage 2: Automating the force method using a matrix approach [4-7] involves constructing the flexibility matrix for a structure [8-17], and constructing equivalent numerical models using the finite element force method [18-29] is known by several names, such as: the integrated force method [21-24], graph-theoretical force method [25,26], loop resultant method [27], generalized flexibility method [28], and base force element method [29]. The use of matrix algebra and matrix calculus as computational tools greatly simplifies the solution of discrete equations in numerical models [30-32]. Instead of solving a problem through manual calculations, digital computers are used primarily to solve equations.

The purpose of this article is to consider three types of finite elements with extended shape forms by the mode shapes of a fixed-fixed rod, and their flexibility and inverse mass matrices respectively. The modes of the following form are used [33]:

In the axial vibration:

$$W_n^{ax}(x,t) = R_n(t)U_n^{ax}(x), \tag{a}$$

where the axial mode shapes $U_n^{ax}(x) = \sin(\frac{n\pi x}{l_e})$, n = 1,2,3,…; and the corresponding time function $R_n(t)$.

In the bending vibration:

$$W_n^{be}(x,t) = S_n(t)U_n^{be}(x), \tag{b}$$

where $U_n^{be}(x) = \{\sin(\frac{\lambda_n x}{l_e}) - sh(\frac{\lambda_n x}{l_e}) - \frac{\sin(\lambda_n) - sh(\lambda_n)}{\cos(\lambda_n) - ch(\lambda_n)}[\cos(\frac{\lambda_n x}{l_e}) - ch(\frac{\lambda_n x}{l_e})]\}$ are the bending mode shapes corresponding to the parameters λ_1 = 4.7300, λ_2 = 7.8532, λ_3 = 10.9956, λ_4 = 14.1372, λ_5 = 17.2788, $\lambda_n = \frac{2n+1}{2}\pi$ (n > 5); and the corresponding time function $S_n(t)$.

Formulations: The differential and the functional forms of vibration in terms of force unknowns

In Fig. 1, an element-rod is subjected to an external distributed longitudinal force q(x) per unit length. The displacement u(x) and internal axial force N(x) represent the response of this ele-ment-rod due to a known applied load.

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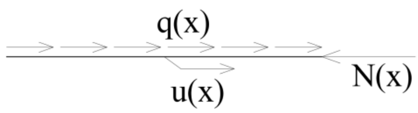

Fig. 1. The tension-compression element-rod

Let us establish the equation of longitudinal vibration of an element-rod.

The following relationships can be established [34]:

$$\frac{1}{EA} N = \varepsilon , \tag{1}$$

and

$$\varepsilon = u' , \tag{2}$$

where EA is the longitudinal stiffness, ε is the axial strain, $(...)'$ denotes a derivative with respect to x.

Substituting expression (2) into expression (1) and taking its derivative with respect to x:

$$\frac{1}{EA} N' = u'' . \tag{3}$$

In addition, we have the following relation:

$$N' + q = \rho \frac{\partial^2 u}{\partial t^2} = \rho \ddot{u} , \tag{4}$$

or

$$N' \frac{1}{\rho} + q \frac{1}{\rho} = \ddot{u} , \tag{5}$$

where ρ is the density of the material, a dot over u denotes a partial derivative with respect to time.

By differentiating Eq. (5) with respect to x, we obtain

$$\frac{\partial}{\partial x}(N' \frac{1}{\rho}) + \frac{\partial}{\partial x}(q \frac{1}{\rho}) = \frac{\partial}{\partial x}(\ddot{u}) , \tag{6}$$

or (if ρ = const(x))

$$N'' \frac{1}{\rho} + q' \frac{1}{\rho} = (u')^{\cdot\cdot} . \tag{7}$$

Substituting Eq. (2) into Eq. (7), we obtain

$$N'' \frac{1}{\rho} + q' \frac{1}{\rho} = \ddot{\varepsilon} . \tag{8}$$

Substituting Eq. (1) into Eq. (8), we obtain

$$\frac{1}{\rho} N'' + q' \frac{1}{\rho} = \frac{1}{EA} \ddot{N} . \tag{9}$$

From Eq. (9), we have the following functional form in terms of the function $N(x,t)$:

$$L(N) = \int_{t_1}^{t_2} \int_{0}^{l_e} \left(\frac{1}{2} \frac{1}{\rho} (N')^2 - \frac{1}{2} \frac{1}{EA} (\dot{N})^2 - \frac{1}{\rho} N q' \right) dx dt . \tag{10}$$

For the longitudinal vibration of a rod, $N(t,x)$ is assumed to be in the form:

$$N(x,t) = N_1(t)(1-\frac{x}{l_e}) + N_2(t)\frac{x}{l_e} = [(1-\frac{x}{l_e}) \quad \frac{x}{l_e}]\begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} = H(x)N(t), \tag{11}$$

where $H(x) = [(1-\frac{x}{l_e}) \quad \frac{x}{l_e}]$, $N(t) = \{N_1(t) \quad N_2(t)\}^T$, the superscript T denotes the transpose of the matrix, and l_e – the initial length of an element rod.

By substituting the expression (11) into each term of the functional form (10), we obtain

$$\int_{t_1}^{t_2}\int_0^{l_e}\left(\frac{1}{2}\frac{1}{\rho}(N')^2\right)dxdt = \int_{t_1}^{t_2}\int_0^{l_e}\left(\frac{1}{2}N^T\frac{1}{\rho}(H')^T H' N\right)dxdt, \tag{12}$$

$$\int_{t_1}^{t_2}\int_0^{l_e}\left(\frac{1}{2}\frac{1}{EA}(\dot{N})^2\right)dxdt = \int_{t_1}^{t_2}\int_0^{l_e}\left(\frac{1}{2}\dot{N}^T\frac{1}{EA}H^T H\dot{N}\right)dxdt, \tag{13}$$

$$\int_{t_1}^{t_2}\int_0^{l_e}\left(\frac{1}{\rho}Nq'\right)dxdt = \int_{t_1}^{t_2}\int_0^{l_e}\left(N^T\frac{1}{\rho}Hq'\right)dxdt. \tag{14}$$

From the expressions (12), (13) and (14), by setting $Z_n^{ax} = \int_0^{l_e}\frac{1}{\rho}(H')^T H' dx$ and it is called the inverse mass matrix (IMM), $V_n^{ax} = \int_0^{l_e}\frac{1}{EA}H^T H dx$ – the flexibility matrix (FM), $P = \int_0^{l_e}\frac{1}{\rho}Hq' dx$. Then, it is not difficult to obtain $Z_0^{ax} = \frac{1}{\rho}\begin{bmatrix} \frac{1}{l_e} & \frac{-1}{l_e} \\ (sym.) & \frac{1}{l_e} \end{bmatrix}$, $V_0^{ax} = \frac{1}{EA}\begin{bmatrix} \frac{l_e}{3} & \frac{l_e}{6} \\ (sym.) & \frac{l_e}{3} \end{bmatrix}$.

The various types of axial element forces can be set for matrices Z_n^{ax} and V_n^{ax} of size $(2+n)\times(2+n)$ by using $H(x) = [(1-\frac{x}{l_e}) \quad \frac{x}{l_e} \quad \sin\frac{\pi x}{l_e} \quad \sin\frac{2\pi x}{l_e} \quad \cdots \quad \sin\frac{n\pi x}{l_e}]$ and $\{N(t)\} = \{N_1(t) \quad N_2(t) \quad R_1(t) \quad R_2(t) \quad \cdots \quad R_n(t)\}^T$, here n = 1,2,….

When n = 1, we have a pair of matrices Z_1^{ax} and V_1^{ax} with size 3×3 as follows:

$$Z_1^{ax} = \frac{1}{\rho}\begin{bmatrix} \frac{1}{l_e} & \frac{-1}{l_e} & \int_0^{l_e}\frac{-\pi}{l_e^2}\cos(\frac{\pi x}{l_e})dx \\ & \frac{1}{l_e} & \int_0^{l_e}\frac{\pi}{l_e^2}\cos(\frac{\pi x}{l_e})dx \\ (sym.) & & \int_0^{l_e}\frac{\pi^2}{l_e^2}\cos^2(\frac{\pi x}{l_e})dx \end{bmatrix}, \quad V_1^{ax} = \frac{1}{EA}\begin{bmatrix} \frac{l_e}{3} & \frac{l_e}{6} & \int_0^{l_e}(1-\frac{x}{l_e})\sin(\frac{\pi x}{l_e})dx \\ & \frac{l_e}{3} & \int_0^{l_e}\frac{x}{l_e}\sin(\frac{\pi x}{l_e})dx \\ (sym.) & & \int_0^{l_e}\sin^2(\frac{\pi x}{l_e})dx \end{bmatrix}.$$

Now, let's rewrite the functional form (10) as below:

$$L(N) = \int_{t_1}^{t_2}\left(\frac{1}{2}N^T Z^{ax} N - \frac{1}{2}\dot{N}^T V^{ax}\dot{N} - N^T P\right)dt. \tag{15}$$

The equation of free longitudinal vibration can be obtained based on the functional form (15):

$$V^{ax}\ddot{N} + Z^{ax}N = 0\,, \tag{16}$$

where $\det(Z^{ax}) = 0$, $\det(V^{ax}) \neq 0$.

Next, consider an element-rod under external transverse forces q(x) as shown in Fig. 2. The displacement v(x), the transverse force Q(x) and the bending moment M(x) represent the response of this element-rod due to applied load q(x).

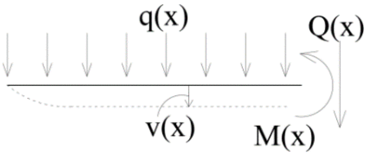

Fig. 2. The bending element-rod: _____ the initial state, …… the deformed state

Let us establish the equation of bending vibration of an element-rod.

The following relationships can be established [34]:

$$v^{//} = \frac{-M}{EI}\,, \tag{17}$$

and

$$Q = M^{/}\,, \tag{18}$$

where EI is the bending stiffness, $(...)^{//}$ is the second derivative of the function.

By setting the second derivative of Eq. (17) with respect to x, we obtain (if EI = const(x))

$$v^{IV} = \frac{-M^{//}}{EI}\,. \tag{19}$$

In addition, we have the equation of motion for displacement v(x):

$$EIv^{IV} + \rho\ddot{v} = q\,. \tag{20}$$

Substituting Eq. (19) into Eq. (20), we obtain

$$-M^{//} + \rho\ddot{v} = q\,, \tag{21}$$

or

$$\frac{-1}{\rho}M^{//} + \ddot{v} = \frac{1}{\rho}q\,. \tag{22}$$

Let's set up the second derivative of Eq. (22) with respect to x, we obtain

$$(\frac{-1}{\rho}M^{//})^{//} + (v^{//})^{\square\square} = \frac{1}{\rho}q^{//}\,. \tag{23}$$

Substituting Eq. (17) into Eq. (23), we obtain

$$\frac{1}{\rho}M^{IV} + \frac{1}{EI}\ddot{M} + \frac{1}{\rho}q^{//} = 0\,. \tag{24}$$

From Eq. (24), the functional form can be expressed in terms of the function $M(x,t)$:

$$L(M) = \int_{t_1}^{t_2}\int_{0}^{l_e}\left(\frac{1}{2}\frac{1}{\rho}(M^{//})^2 - \frac{1}{2}\frac{1}{EI}\dot{M}^2 + \frac{1}{\rho}Mq^{//}\right)dxdt\,. \tag{25}$$

Assuming the function $M(x,t)$ for the bending vibration of a rod as

$$M(x,t) = Q_1(t)\text{Э}_1(x) + M_1(t)\text{Э}_2(x) + Q_2(t)\text{Э}_3(x) + M_2(t)\text{Э}_4(x)\,, \tag{26}$$

where $Э_i(x)$ – the Hermite polynomials ($i = 1,2,3,4$).

The expression (26) can be represented in matrix form:

$$M(x,t) = [Э_1(x) \quad Э_2(x) \quad Э_3(x) \quad Э_4(x)] \begin{Bmatrix} Q_i(t) \\ M_i(t) \end{Bmatrix} = HF\,, \qquad (27)$$

where $H = [Э_1(x) \quad Э_2(x) \quad Э_3(x) \quad Э_4(x)]$ and $F = \begin{Bmatrix} Q_i(t) \\ M_i(t) \end{Bmatrix} = \{Q_1(t) \quad M_1(t) \quad Q_2(t) \quad M_2(t)\}^T$.

By substituting the expression (27) into each term of the functional form (25), we have

$$\int_{t_1}^{t_2}\int_0^{l_e}\left(\frac{1}{2}\frac{1}{\rho}(M^{//})^2\right)dxdt = \int_{t_1}^{t_2}\int_0^{l_e}\left(\frac{1}{2}F^T\frac{1}{\rho}(H^{//})^T H^{//}F\right)dxdt\,, \qquad (28)$$

$$\int_{t_1}^{t_2}\int_0^{l_e}\left(\frac{1}{2}\frac{1}{EI}\dot{M}^2\right)dxdt = \int_{t_1}^{t_2}\int_0^{l_e}\left(\frac{1}{2}\dot{F}^T\frac{1}{EI}H^T H\dot{F}dx\right)dxdt\,, \qquad (29)$$

$$\int_{t_1}^{t_2}\int_0^{l_e}\left(\frac{1}{\rho}Mq^{//}\right)dxdt = \int_{t_1}^{t_2}\int_0^{l_e}\left(F^T\frac{1}{\rho}Hq^{//}\right)dxdt\,. \qquad (30)$$

From the expressions (28), (29) and (30) by setting $Z_n^{be} = \int_0^{l_e}\frac{1}{\rho}(H^{//})^T H^{//}dx$ – the IMM, $V_n^{be} = \int_0^{l_e}\frac{1}{EI}H^T H dx$ – the FM, and $P = \int_0^{l_e}\frac{1}{\rho}Hq^{//}dx$, we obtain

$$Z_0^{be} = \frac{1}{\rho}\begin{bmatrix} \frac{12}{l_e^3} & \frac{6}{l_e^2} & \frac{-12}{l_e^3} & \frac{6}{l_e^2} \\ & \frac{4}{l_e} & \frac{-6}{l_e^2} & \frac{2}{l_e} \\ & (sym.) & \frac{12}{l_e^3} & \frac{-6}{l_e^2} \\ & & & \frac{4}{l_e} \end{bmatrix},\quad V_0^{be} = \frac{1}{EI}\begin{bmatrix} \frac{13l_e}{35} & \frac{11l_e^2}{210} & \frac{9l_e}{70} & \frac{-13l_e^2}{420} \\ & \frac{l_e^3}{105} & \frac{13l_e^2}{420} & \frac{-l_e^3}{140} \\ & (sym.) & \frac{13l_e}{35} & \frac{-11l_e^2}{210} \\ & & & \frac{l_e^3}{105} \end{bmatrix}.$$

The various types of bending element forces can be set for matrices Z_n^{be} and V_n^{be} of size $(4+n)\times(4+n)$ by using $H = [Э_1(x) \quad Э_2(x) \quad Э_3(x) \quad Э_4(x) \quad U_1^{be}(x) \quad U_2^{be}(x) \quad \cdots \quad U_n^{be}(x)]$ and $F = \{Q_1(t) \quad M_1(t) \quad Q_2(t) \quad M_2(t) \quad S_1(t) \quad S_2(t) \quad \cdots \quad S_n(t)\}^T$, n = 1,2,…

For n = 1, we have a pair of matrices Z_1^{be} and V_1^{be} with size 5×5 as follows:

$$Z_1^{be} = \frac{1}{\rho}\begin{bmatrix} \frac{12}{l_e^3} & \frac{6}{l_e^2} & \frac{-12}{l_e^3} & \frac{6}{l_e^2} & \int_0^{l_e} Э_1^{//}(U_1^{be})^{//}dx \\ & \frac{4}{l_e} & \frac{-6}{l_e^2} & \frac{2}{l_e} & \int_0^{l_e} Э_2^{//}(U_1^{be})^{//}dx \\ & & \frac{12}{l_e^3} & \frac{-6}{l_e^2} & \int_0^{l_e} Э_3^{//}(U_1^{be})^{//}dx \\ & (sym.) & & \frac{4}{l_e} & \int_0^{l_e} Э_4^{//}(U_1^{be})^{//}dx \\ & & & & \int_0^{l_e}(U_1^{be})^{//}(U_1^{be})^{//}dx \end{bmatrix},\quad V_1^{be} = \frac{1}{EI}\begin{bmatrix} \frac{13l_e}{35} & \frac{11l_e^2}{210} & \frac{9l_e}{70} & \frac{-13l_e^2}{420} & \int_0^{l_e} Э_1 U_1^{be}dx \\ & \frac{l_e^3}{105} & \frac{13l_e^2}{420} & \frac{-l_e^3}{140} & \int_0^{l_e} Э_2 U_1^{be}dx \\ & & \frac{13l_e}{35} & \frac{-11l_e^2}{210} & \int_0^{l_e} Э_3 U_1^{be}dx \\ & (sym.) & & \frac{l_e^3}{105} & \int_0^{l_e} Э_4 U_1^{be}dx \\ & & & & \int_0^{l_e} U_1^{be}U_1^{be}dx \end{bmatrix}.$$

We can rewrite the functional form (25) as

$$L(M)=\int_{t_1}^{t_2}\left(\frac{1}{2}F^T Z^{be}F-\frac{1}{2}\dot{F}^T V^{be}\dot{F}+F^T P\right)dt\,. \tag{31}$$

The equation of free bending vibration can be obtained based on the functional form (31):

$$V^{be}\ddot{F}+Z^{be}F=0\,, \tag{32}$$

where $\det(Z^{be})=0$, $\det(V^{be})\neq 0$.

The equation of natural oscillation of a plane element is shown as below:

$$V^{pl}\ddot{F}+Z^{pl}F=0\,, \tag{33}$$

where $\det(Z^{pl})=0$, $\det(V^{pl})\neq 0$, and for n = 0:

$$Z_0^{pl}=\frac{1}{\rho}\begin{pmatrix} \frac{1}{l_e} & 0 & 0 & -\frac{1}{l_e} & 0 & 0 \\ & \frac{12}{l_e^3}r_y^2 & \frac{6}{l_e^2}r_y^2 & 0 & -\frac{12}{l_e^3}r_y^2 & \frac{6}{l_e^2}r_y^2 \\ & & \frac{4}{l_e}r_y^2 & 0 & -\frac{6}{l_e^2}r_y^2 & \frac{2}{l_e}r_y^2 \\ & & & \frac{1}{l_e} & 0 & 0 \\ & (sym.) & & & \frac{12}{l_e^3}r_y^2 & -\frac{6}{l_e^2}r_y^2 \\ & & & & & \frac{4}{l_e}r_y^2 \end{pmatrix},\quad V_0^{pl}=\frac{1}{EA}\begin{pmatrix} \frac{l_e}{3} & 0 & 0 & \frac{l_e}{6} & 0 & 0 \\ & \frac{13l_e}{35} & \frac{11l_e^2}{210} & 0 & \frac{9l_e}{70} & -\frac{13l_e^2}{420} \\ & & \frac{l_e^3}{105} & 0 & \frac{13l_e^2}{420} & -\frac{l_e^3}{140} \\ & & & \frac{l_e}{3} & 0 & 0 \\ & (sym.) & & & \frac{13l_e}{35} & -\frac{11l_e^2}{210} \\ & & & & & \frac{l_e^3}{105} \end{pmatrix},\ \text{here } r_y^2=\frac{I_y}{A}\,.$$

The equation of natural oscillation of a space element is shown as below:

$$V^{sp}\ddot{F}+Z^{sp}F=0\,, \tag{34}$$

where $\det(Z^{sp})=0$, $\det(V^{sp})\neq 0$, and for n = 0.

Numerical results of structural rods

Example 1: Consider a rod with different boundary conditions.

Initial data: Young's modulus E = 2.1×1011 (N/m2), the length of a rod L = 1 (m), the square cross-section with a square side 0.1 (m) and the mass per unit length of a rod ρ = 7830 (kg/m3).

The exact solutions evaluated using the differential equation of vibration of a rod. The n-th natural frequency of longitudinal vibration of a rod with fixed-free ends is given by $\omega_n=\frac{(2n+1)\pi}{2L}\sqrt{\frac{E}{\rho}}$, here $n=0,1,2,\ldots$; the n-th natural frequency of bending vibration of a rod with fixed-fixed ends is given by $\omega_n=\left(\frac{\lambda_n}{L}\right)^2\sqrt{\frac{EI}{\rho A}}$, here λ_1 = 4.7300, λ_2 = 7.8532, λ_3 = 10.9956, λ_4 = 14.1372, λ_5 = 17.2788, $\lambda_n=\frac{\pi}{2}(2n+1)$, $(n>5)$.

The numerical results of the first five natural frequencies are performed for rods from 1 to 20 finite elements (FE) using the developed program with the help of Matlab software. The normal, italic and bold values are obtained from the first type $(n=0)$, second type $(n=1)$ and third type $(n=2)$ of element forces, respectively.

In figures 3 and 4, numerical results of the first two natural frequencies using different types of element forces are compared with exact solutions.

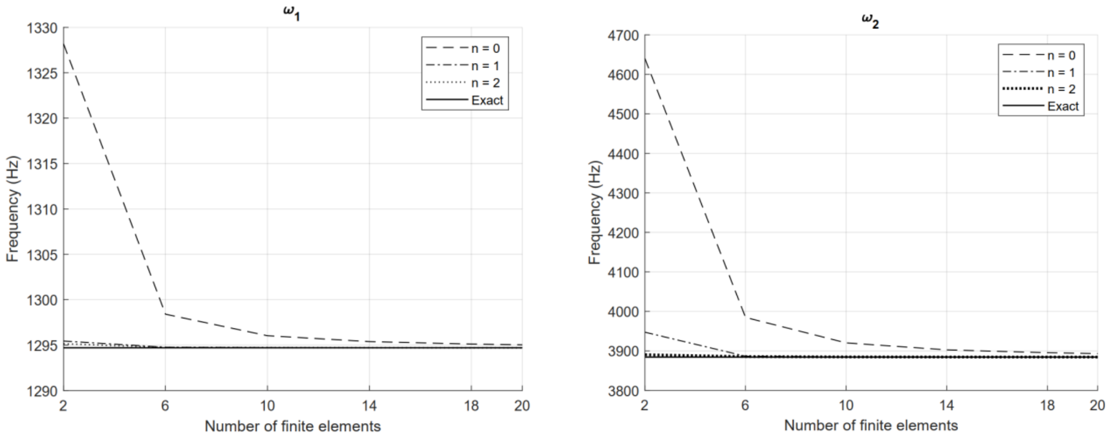

Fig. 3. The first (left) and second (right) natural frequencies of axial vibration

The error between numerical and exact solutions can be calculated by

$$\delta_i = \frac{|Numerical - Exact|}{Exact}. \qquad (35)$$

The errors of numerical results are generated by three types of element forces and given in tables 1 and 2.

Table 1

The errors of the longitudinal natural vibration frequencies

δ_i	2FE	6FE	10FE	14FE	18FE	20FE	n
1	0.0259	0.0029	0.0010	0.0005	0.0003	0.0003	0
	0.0006	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*1*
	0.0003	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**2**
2	0.1946	0.0259	0.0093	0.0047	0.0029	0.0023	0
	0.0162	*0.0006*	*0.0002*	*0.0001*	*0.0000*	*0.0000*	*1*
	0.0016	**0.0003**	**0.0001**	**0.0001**	**0.0000**	**0.0000**	**2**
3	-	0.0719	0.0259	0.0132	0.0080	0.0064	0
	-	*0.0025*	*0.0006*	*0.0002*	*0.0001*	*0.0001*	*1*
	-	**0.0008**	**0.0003**	**0.0002**	**0.0001**	**0.0001**	**2**
4	-	0.1365	0.0508	0.0259	0.0156	0.0126	0
	-	*0.0073*	*0.0015*	*0.0006*	*0.0003*	*0.0002*	*1*
	-	**0.0012**	**0.0006**	**0.0003**	**0.0002**	**0.0002**	**2**
5	-	0.1946	0.0837	0.0428	0.0259	0.0209	0
	-	*0.0162*	*0.0032*	*0.0012*	*0.0006*	*0.0004*	*1*
	-	**0.0016**	**0.0009**	**0.0005**	**0.0003**	**0.0003**	**2**

From table 1 and figure 3, we can draw the following facts:

– The rate of convergence of frequencies of the first type $(n=0)$ is slower as compared to the second $(n=1)$ and third $(n=2)$ types (e.g., values $\delta_{1(n=0)} = 0.0259$, $\delta_{1(n=1)} = 0.0006$, $\delta_{1(n=2)} = 0.0003$ by using 2 FE).
– The rate of convergence of the third $(n=2)$ type is much fast as compared to the first $(n=0)$ and second $(n=1)$ types for most frequencies (e.g., values $\delta_{2(n=2)} = 0.0016$, $\delta_{2(n=1)} = 0.0162$, $\delta_{2(n=0)} = 0.1946$ by using 2 FE).

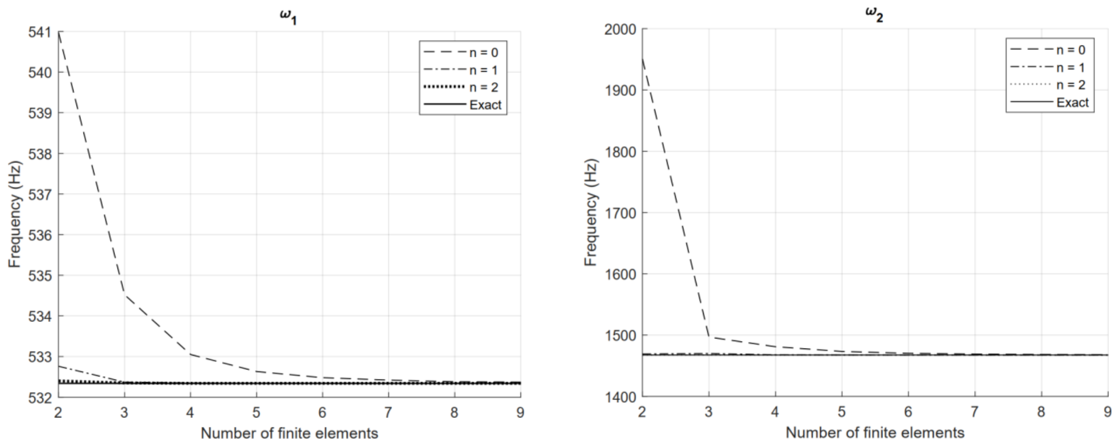

Fig. 4. The first (left) and second (right) natural frequencies of bending vibration

Table 2

The errors of the natural frequencies of bending vibration

δ_i	2FE	3FE	4FE	5FE	6FE	7FE	8FE	9FE	n
1	0.0162	0.0041	0.0013	0.0005	0.0003	0.0001	0.0001	0.0001	0
	0.0008	*0.0001*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*1*
	0.0001	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**2**
2	0.3292	0.0200	0.0092	0.0040	0.0020	0.0011	0.0006	0.0004	0
	0.0012	*0.0017*	*0.0003*	*0.0001*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*1*
	0.0001	**0.0001**	**0.0001**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**2**
3	-	0.2101	0.0214	0.0138	0.0072	0.0040	0.0024	0.0015	0
	-	*0.0013*	*0.0024*	*0.0005*	*0.0002*	*0.0001*	*0.0000*	*0.0000*	*1*
	-	**0.0001**	**0.0002**	**0.0001**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**2**
4	-	0.4548	0.1689	0.0218	0.0175	0.0103	0.0062	0.0040	0
	-	*0.0192*	*0.0013*	*0.0029*	*0.0008*	*0.0003*	*0.0002*	*0.0001*	*1*
	-	**0.0005**	**0.0001**	**0.0002**	**0.0001**	**0.0001**	**0.0000**	**0.0000**	**2**
5	-	-	0.2942	0.1485	0.0219	0.0203	0.0131	0.0085	0
	-	-	*0.0129*	*0.0014*	*0.0033*	*0.0010*	*0.0005*	*0.0002*	*1*
	-	-	**0.0003**	**0.0001**	**0.0002**	**0.0001**	**0.0001**	**0.0000**	**2**

The following facts can be drawn in table 2 and figure 4:

– The rate of convergence of frequencies of the first type $(n=0)$ is much slower as compared to the second $(n=1)$ and third $(n=2)$ types for most cases from 2 to 9 FE (e.g., values $\delta_{2(n=0)}=0.3292$, $\delta_{2(n=1)}=0.0012$, $\delta_{2(n=2)}=0.0001$ by using 2 FE).

– The rate of convergence of frequencies of the third type $(n=2)$ is fast as compared to the second $(n=1)$ and first $(n=0)$ types (e.g., values $\delta_{4(n=2)}=0.0005$, $\delta_{4(n=1)}=0.0192$, $\delta_{4(n=0)}=0.4548$ by using 3 FE).

Numerical results of structural frames

Example 2. The planar frame is given, see Fig. 5.

Initial data: The elastic modulus $E = 2\times10^8$ (kN/m^2), the cross-sectional area $A = 0.0144$ (m^2), the moment of inertia $I = 1.728\times10^{-5}$ (m^4), the mass per unit length of a rod $\rho = 2750$ (kg/m^3).

Let's consider three options for the finite element mesh: 3, 6 and 9 finite elements.

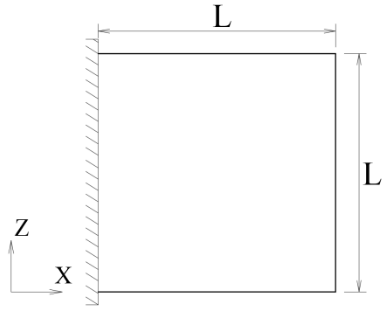

Fig. 5. The planar frame, 3 rods

Table 3 shows the natural frequencies (Hz) taking into account the axial and bending vibrations, the error of results is enclosed in brackets.

Table 3

The first five natural frequencies of the planar frame

ω_i	3FE			6FE			9FE			Exact
	n = 0	n = 1	n = 2	n = 0	n = 1	n = 2	n = 0	n = 1	n = 2	
1	150.49	*150.19*	**150.18**	150.21	*150.17*	**150.17**	150.18	*150.17*	**150.17**	145.65
	(0.033)	*(0.031)*	**(0.031)**	(0.031)	*(0.031)*	**(0.031)**	(0.031)	*(0.031)*	**(0.031)**	
2	682.32	*578.57*	**578.35**	581.84	*578.37*	**578.32**	579.02	*578.30*	**578.30**	553.92
	(0.231)	*(0.045)*	**(0.044)**	(0.050)	*(0.044)*	**(0.044)**	(0.045)	*(0.044)*	**(0.044)**	
3	1505.7	*968.60*	**966.66**	978.31	*967.22*	**966.68**	969.67	*966.66*	**966.62**	898.52
	(0.675)	*(0.078)*	**(0.076)**	(0.088)	*(0.076)*	**(0.076)**	(0.079)	*(0.076)*	**(0.076)**	
4	1803.9	*991.66*	**991.50**	1006.7	*992.05*	**991.59**	995.10	*991.53*	**991.50**	909.85
	(0.982)	*(0.090)*	**(0.090)**	(0.106)	*(0.090)*	**(0.090)**	(0.093)	*(0.090)*	**(0.090)**	
5	1923.7	*1727.9*	**1629.6**	1695.9	*1628.4*	**1627.9**	1641.7	*1627.8*	**1627.7**	1554.4
	(0.237)	*(0.112)*	**(0.048)**	(0.091)	*(0.048)*	**(0.047)**	(0.056)	*(0.047)*	**(0.047)**	

Note: The exact solutions are performed by SCAD with 30 FE.

Example 3. Consider the space frame as shown in Fig. 6.

Initial data: The elastic modulus E = 3×107 (kN/m2), the cross-sectional area A = 0.01 (m2), the coefficient of cross-sectional shape κ = 5/6, Poisson's ratio ν = 0.2, the moment of inertia I = 8.33×10-6 (m4), the mass per unit length of a rod ρ = 24517 (N/m3).

Let's consider three options for the finite element mesh: 8, 16 and 24 finite elements.

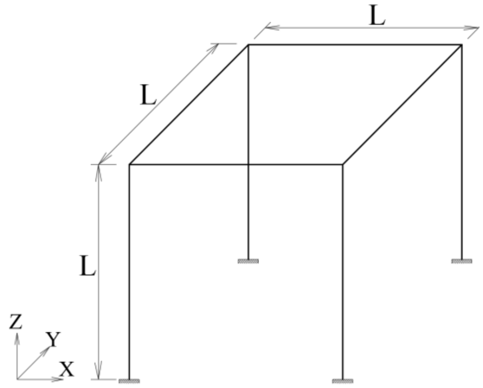

Fig. 6. The space frame, 8 rods

Table 4 shows the natural frequencies (Hz) taking into account the axial, bending and torsional vibrations.

Table 4

The first five natural frequencies of the space frame

ω_i	8FE			16FE			24FE			Exact
	n = 0	n = 1	n = 2	n = 0	n = 1	n = 2	n = 0	n = 1	n = 2	
1	40.064	*39.935*	**39.923**	39.972	*39.963*	**39.963**	39.967	*39.965*	**39.964**	39.419
	(0.016)	*(0.013)*	**(0.013)**	(0.014)	*(0.014)*	**(0.014)**	(0.014)	*(0.014)*	**(0.014)**	
2	50.932	*50.837*	**50.821**	50.852	*50.841*	**50.840**	50.843	*50.840*	**50.840**	50.191
	(0.014)	*(0.013)*	**(0.013)**	(0.013)	*(0.013)*	**(0.013)**	(0.013)	*(0.013)*	**(0.013)**	
3	88.196	*87.726*	**87.634**	87.777	*87.718*	**87.714**	87.724	*87.709*	**87.708**	86.526
	(0.019)	*(0.014)*	**(0.013)**	(0.014)	*(0.014)*	**(0.014)**	(0.014)	*(0.014)*	**(0.014)**	
4	185.47	*164.28*	**163.33**	165.99	*165.40*	**165.34**	165.42	*165.30*	**165.29**	163.84
	(0.132)	*(0.003)*	**(0.003)**	(0.013)	*(0.009)*	**(0.009)**	(0.010)	*(0.009)*	**(0.009)**	
5	225.36	*200.53*	**199.28**	193.87	*193.32*	**193.25**	192.96	*192.80*	**192.79**	189.81
	(0.187)	*(0.056)*	**(0.050)**	(0.021)	*(0.018)*	**(0.018)**	(0.017)	*(0.016)*	**(0.016)**	

Note: The exact solutions are performed by SCAD with 80 FE.

From tables 3 and 4, we can see that when using 3FE for the planar frame and 8FE for the space frame, the frequencies obtained by the third type $(n = 2)$ of element forces, which show high accuracy even with high-order frequencies (e.g., values $\delta_{5(n=2)} = 0.048$, $\delta_{5(n=1)} = 0.112$, $\delta_{5(n=0)} = 0.237$ in table 3; values $\delta_{5(n=2)} = 0.050$, $\delta_{5(n=1)} = 0.056$, $\delta_{5(n=0)} = 0.187$ in table 4).

Conclusions

Present formulations provide frequencies of structural rods and frames with the use of flexibility matrix and inverse mass matrix using the mode shapes of a fixed-fixed rod. From above results, we concluded that increasing in parameters $n = 1$ and $n = 2$ correspond to first and second modes, results the decrease in the rate of convergence of frequencies as shown in tables 1, 2, 3 and 4. The variation in frequency is sudden decrement between parameters $n = 0$ and $n = 1$ (e.g., values $\delta_{1(n=0)} = 0.0259$, $\delta_{1(n=1)} = 0.0006$ in table 1 or values $\delta_{4(n=0)} = 0.132$, $\delta_{4(n=1)} = 0.003$ in table 4), i.e., calculations using the element forces in the form of the finite element force method give fairly accurate results even with a coarse mesh.

APPENDIX

$$Z_0^{sp} = \frac{1}{\rho}\begin{bmatrix}
\frac{1}{l_e} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{l_e} & 0 & 0 & 0 & 0 & 0 \\
 & \frac{12}{l_e^3}r_z^2 & 0 & 0 & 0 & \frac{6}{l_e^2}r_z^2 & 0 & -\frac{12}{l_e^3}r_z^2 & 0 & 0 & 0 & \frac{6}{l_e^2}r_z^2 \\
 & & \frac{12}{l_e^3}r_y^2 & 0 & -\frac{6}{l_e^2}r_y^2 & 0 & 0 & 0 & -\frac{12}{l_e^3}r_y^2 & 0 & -\frac{6}{l_e^2}r_y^2 & 0 \\
 & & & \frac{1}{2l_e(1+\nu)}r_x^2 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2l_e(1+\nu)}r_x^2 & 0 & 0 \\
 & & & & \frac{4}{l_e}r_y^2 & 0 & 0 & 0 & \frac{6}{l_e^2}r_y^2 & 0 & \frac{2}{l_e}r_y^2 & 0 \\
 & & & & & \frac{4}{l_e}r_z^2 & 0 & -\frac{6}{l_e^2}r_z^2 & 0 & 0 & 0 & \frac{2}{l_e}r_z^2 \\
 & & & & & & \frac{1}{l_e} & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & \frac{12}{l_e^3}r_z^2 & 0 & 0 & 0 & -\frac{6}{l_e^2}r_z^2 \\
 & & & & & & & & \frac{12}{l_e^3}r_y^2 & 0 & \frac{6}{l_e^2}r_y^2 & 0 \\
 & (sym.) & & & & & & & & \frac{1}{2l_e(1+\nu)}r_x^2 & 0 & 0 \\
 & & & & & & & & & & \frac{4}{l_e}r_y^2 & 0 \\
 & & & & & & & & & & & \frac{4}{l_e}r_z^2
\end{bmatrix},$$

$$V_0^{sp} = \frac{1}{EA}\begin{bmatrix}
\frac{l_e}{3} & 0 & 0 & 0 & 0 & 0 & \frac{l_e}{6} & 0 & 0 & 0 & 0 & 0 \\
 & \frac{13l_e}{35} & 0 & 0 & 0 & \frac{11l_e^2}{210} & 0 & \frac{9l_e}{70} & 0 & 0 & 0 & -\frac{13l_e^2}{420} \\
 & & \frac{13l_e}{35} & 0 & -\frac{11l_e^2}{210} & 0 & 0 & 0 & \frac{9l_e}{70} & 0 & \frac{13l_e^2}{420} & 0 \\
 & & & \frac{r_x^2}{3}l_e & 0 & 0 & 0 & 0 & 0 & \frac{r_x^2}{6}l_e & 0 & 0 \\
 & & & & \frac{l_e^3}{105} & 0 & 0 & 0 & -\frac{13l_e^2}{420} & 0 & -\frac{l_e^3}{140} & 0 \\
 & & & & & \frac{l_e^3}{105} & 0 & \frac{13l_e^2}{420} & 0 & 0 & 0 & -\frac{l_e^3}{140} \\
 & & & & & & \frac{l_e}{3} & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & \frac{13l_e}{35} & 0 & 0 & 0 & -\frac{11l_e^2}{210} \\
 & & & & & & & & \frac{13l_e}{35} & 0 & \frac{11l_e^2}{210} & 0 \\
 & (sym.) & & & & & & & & \frac{r_x^2}{3}l_e & 0 & 0 \\
 & & & & & & & & & & \frac{l_e^3}{105} & 0 \\
 & & & & & & & & & & & \frac{l_e^3}{105}
\end{bmatrix},$$

here $r_x^2 = \frac{I_x}{A}$, $r_z^2 = \frac{I_z}{A}$..

ВКЛАД АВТОРОВ | CONTRIBUTION OF THE AUTHORS

Х.Х. Нго, В.В. Лалин, Т.К.Ч. Ле – анализ и интерпретация результатов; подготовка и редактирование текста. Все авторы прочитали и одобрили окончательный вариант рукописи.

H.H. Ngo, V.V. Lalin, T.Q.T. Le – analysis and interpretation of results; draft manuscript preparation. All authors reviewed the results and approved the final version of the manuscript.

КОНФЛИКТ ИНТЕРЕСОВ | CONFLICT OF INTEREST

Авторы заявляют об отсутствии конфликта интересов.

The authors declare no conflict of interest.

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ИНФОРМАЦИЯ ОБ АВТОРАХ | INFORMATION ABOUT THE AUTHORS

Нго Хыу Хиеу – аспирант, Санкт-Петербургский политехнический университет Петра Великого, Санкт-Петербург, Российская Федерация, hieupolytech1993@gmail.com, https://orcid.org/0000-0002-7882-7062

Ngo Huu Hieu, Postgraduate Student, Peter the Great St. Petersburg Polytechnic University, Saint Petersburg, Russian Federation, hieupolytech1993@gmail.com, https://orcid.org/0000-0002-7882-7062

Лалин Владимир Владимирович – доктор технических наук, профессор, Санкт-Петербургский политехнический университет Петра Великого, Санкт-Петербург, Российская Федерация, vllalin@yandex.ru, https://orcid.org/0000-0003-3850-424X

Lalin Vladimir Vladimirovich, Doctor of Engineering Sciences, Professor, Peter the Great St. Petersburg Polytechnic University, Saint Petersburg, Russian Federation, vllalin@yandex.ru, https://orcid.org/0000-0003-3850-424X

Ле Ты Куанг Чунг – кандидат технических наук, Вьетнамский институт атомной энергии, Министерство науки и технологий, Ханой, Вьетнам, quangtrung1690@gmail.com, https://orcid.org/0000-0002-6547-4632

Le Tu Quang Trung, PhD in Sci. Tech., Vietnam Atomic Energy Institute, Ministry of Science and Technology (Hanoi, Vietnam), quangtrung1690@gmail.com, https://orcid.org/0000-0002-6547-4632

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